(Anti)de Sitter/Poincaré symmetries and representations from Poincaré/Galilei through a classical deformation approach

Francisco J Herranz[1] **and Mariano Santander**[2]

[1] Departamento de Física, Escuela Politécnica Superior, Universidad de Burgos, E–09001 Burgos, Spain
[2] Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, E–47011 Valladolid, Spain

Abstract

A classical deformation procedure, based on universal enveloping algebras, Casimirs and curvatures of symmetrical homogeneous spaces, is applied to several cases of physical relevance. Starting from the $(3 + 1)$D Galilei algebra, we describe at the level of representations the process leading to its two physically meaningful deformed neighbours. The Poincaré algebra is obtained by introducing a negative curvature in the flat Galilean phase space (or space of worldlines), while keeping a flat spacetime. To be precise, starting from a representation of the Galilei algebra with both Casimirs *different* from zero, we obtain a representation of the Poincaré algebra with both Casimirs *necessarily* equal to zero. The Poincaré angular momentum, Pauli–Lubanski components, position and velocity operators, etc are expressed in terms of 'Galilean' operators through some expressions deforming the proper Galilean ones. Similarly, the Newton–Hooke algebras appear by endowing spacetime with a nonzero curvature, while keeping a flat phase space. The same approach, starting from the $(3 + 1)$D Poincaré algebra provides representations of the (anti)de Sitter as Poincaré deformations.

PACS numbers: 02.20.Sv, 02.20.Qs, 11.30.−j

1. Introduction

Classical deformations of Lie algebras can be considered as the opposite process to Lie algebra contractions. In general, starting from a Lie algebra a *contraction* gives rise to another *more Abelian* algebra by making some structure constants vanish [1–7], while a *deformation* goes to another *less Abelian* algebra 'producing' some nonzero structure constants [7–14]. The idea of contractions of Lie algebras and groups historically appeared in relation with

the non-relativistic limit which brings relativistic mechanics (Poincaré group) to classical mechanics (Galilei group) [1, 2, 15]. In the framework of kinematical algebras the scheme of contractions is well known [16, 17]: starting from the (anti)de Sitter algebra a sequence of different contractions leads to Poincaré, Newton–Hooke, Galilei, ..., ending up at the last stage in the so-called static algebra. From the viewpoint of the graded contraction theory [4, 5], the $(3 + 1)$D case has been studied in [18], and kinematical contractions in arbitrary dimension have been obtained in [19].

The theory of deformations [8, 10, 12] has been mainly developed in connection with cohomology techniques which relate Lie algebra cohomology groups to Lie algebra deformations [11]. This procedure shows in a systematic way which nonzero Lie brackets can be set different from zero for a given Lie algebra leading to all its possible deformations. By following this approach all the $(3 + 1)$D Galilean deformations were obtained in [20]. Recently, classical deformations (within this type of procedures) and contractions have been studied altogether in [21] (see also references therein).

In another approach to deformations, one tries to obtain explicit realizations of a deformed Lie algebra in terms of the universal enveloping algebra of the initial one [7, 9], but as yet these have not been fully systematized and there is no general constructive theory. The procedure given in [7, 9] provides the deformations from the universal enveloping algebra of the inhomogeneous pseudo-orthogonal algebra $\mathfrak{iso}(p, q)$ to the semisimple ones $\mathfrak{so}(p+1, q)$ (with $p + q = N$); these contain as particular cases those deformations starting from the Euclidean algebra and leading to either the elliptic or hyperbolic ones as well as the deformations from Poincaré to both de Sitter algebras. A different method [13] enables one to perform the deformations $\mathfrak{t}_{qp}(\mathfrak{so}(p) \oplus \mathfrak{so}(q)) \to \mathfrak{so}(p, q)$. Similar deformations for unitary algebras can be found in the above references.

In [14], we proposed another deformation method which was formulated in algebraic terms and based, like the above procedures, on the universal enveloping algebra of initial Lie algebras. Nevertheless, the ideas of [14] follow from geometrical considerations which require one to control the deformation by a parameter which has a precise meaning: it is the curvature of some (family of) homogeneous spaces associated with the initial and the deformed Lie algebras. The procedure rests on their (quadratic) Casimir operators. This method was applied to the deformations within the set of $(2 + 1)$D kinematical algebras [16], where the Lie algebras are contractions of $\mathfrak{so}(4)$. In arbitrary dimension, $\mathfrak{so}(N + 1)$ and its kinematical contractions have as many Casimirs as its rank [22]; only one is quadratic and the others are higher-order polynomials in the generators. Then it could happen that a method working for $\mathfrak{so}(4)$ (the only particular case where the two Casimirs are quadratic) may not be directly extendible to higher dimensions.

In this paper, we propose a more general deformation procedure, again based on geometrical ideas (curvatures of homogeneous spaces and Casimirs) which, in principle, can be applied to any Lie algebra in any dimension (not only a kinematical one). Such a method, presented in the following section, leads naturally to some operators fulfilling the deformed commutation rules as in [11, 20, 21], but it also provides explicit expressions of the deformed generators as elements of the universal enveloping algebra of the initial Lie algebra. Thus the resulting expressions may allow a further study of deformations for both operators and representations.

In particular, we apply this procedure to the Galilei algebra in the realistic kinematical dimension, the $(3 + 1)$D case. Hence, the basics of the Galilei algebra structure and its main associated symmetric homogeneous spaces (spacetime and phase space) are recalled in section 3. This allows us to produce, in an explicit constructive way, *two* physically relevant deformations. One reverses the well-known non-relativistic limit by obtaining the Poincaré

algebra expressed through the universal enveloping Galilei one. Furthermore some Poincaré operators such as Pauli–Lubanski components, angular momentum, position and velocity operators, etc are expressed in terms of Galilean ones. We remark that these deformed objects are not provided by cohomological procedures as in the Galilean deformations studied in [20]. The other deformation reverses the ordinary zero-curvature limit in the non-relativistic Newton–Hooke spacetimes, whose generators are obtained within the universal enveloping algebra of the centrally extended Galilei algebra. Deformations of representations for both Newton–Hooke algebras are also presented. These two deformations are explicitly developed in sections 4 and 5, respectively.

Our approach also cover some known cases, for example, the deformations starting from the Poincaré algebra, and leading to both (anti)de Sitter algebras [7, 9] and additionally yielding explicitly (anti)de Sitter representations from Poincaré ones. This is described in section 6. Finally, some remarks and relationships with quantum deformations are pointed out in section 7.

2. A classical deformation method

The Lie algebra deformation method proposed in [14] for the $(2 + 1)$D kinematical algebras [16, 17] will be extended here in order to apply to any Lie algebra of arbitrary dimension.

Let $\mathfrak{g}$ be a Lie algebra which is obtained through a contraction from another Lie algebra $\mathfrak{g}'$, $\mathfrak{g}' \rightarrow \mathfrak{g}$, with Lie groups G and G'. Suppose that the contraction corresponds to the vanishing of the *curvature κ* of some non-flat symmetrical homogeneous space G'/H' and thus leads to a *flat* space G/H, where H and H' are the Lie subgroups of the (common) subalgebras invariant under the contraction, and where both homogeneous spaces are of the same *rank r* (defined by the number of independent invariants of a pair of points in the space under the action of the group [7]; this definition holds for spaces associated with both semisimple and non-semisimple Lie groups; see [23] for the Euclidean space).

Next assume that both $\mathfrak{g}'$ and $\mathfrak{g}$ have m functionally independent Casimirs ($m \geqslant r$). We shall denote by $\mathcal{C}_l$ ($l = 1, \ldots, m$) a linear basis of the centre of the universal enveloping algebra of the contracted Lie algebra $\mathfrak{g}$ (with $\kappa = 0$) and by $\mathcal{C}'_l$ the corresponding one for the deformed Lie algebra $\mathfrak{g}'$ (with $\kappa \neq 0$); the numbering ($l = 1, \ldots, m$) in the set of Casimirs corresponds to increasing order of the $\mathcal{C}_l$ as polynomials in the generators, starting from the quadratic one $\mathcal{C}_1$ (related to the Killling–Cartan form). Although the process involves the choice of a basis, the end result will turn out to be independent of this choice. The deformation process starts from the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and looks for a deformed algebra $\mathfrak{g}'$ by the following four steps:

(1) Writing the first r deformed Casimirs as polynomials in the curvature κ we aim to recover:

$$\mathcal{C}'_s = \mathcal{C}_s + \kappa \mathcal{J}_s^{(1)} + \kappa^2 \mathcal{J}_s^{(2)} + \kappa^3 \mathcal{J}_s^{(3)} + \cdots \qquad s = 1, \ldots, r, \qquad (2.1)$$

where $\mathcal{C}_s$, $\mathcal{J}_s^{(k)}$ are *independent* of κ; the $\mathcal{C}_s$ are just the contracted Casimirs.

(2) Working in $U(\mathfrak{g})$ within an irreducible representation, consider as a 'deformation seed' some linear combination of the r terms *linear* in the curvature κ:

$$\mathcal{J} = \sum_{s=1}^{r} \alpha_s \mathcal{J}_s^{(1)}, \qquad (2.2)$$

where α_s are constants to be determined. Note that any (linear, compatible with the numbering criterion) change in the initial choice of the $\mathcal{C}'_s$ will lead to the same 'seed' because each of the terms linear in κ in the new $\mathcal{C}'_s$ up to rth are themselves a linear

3

combination of the former; this means that the method is intrinsic, even if it is formulated using a particular basis of the centre of the universal enveloping algebra.

(3) For each initial generator X_i of $\mathfrak{g}$, define the deformed generators X_i' of $\mathfrak{g}'$ as the elements in $U(\mathfrak{g})$ given by

$$X_i' := \begin{cases} X_i & \text{if} \quad [\mathcal{J}, X_i] = 0 \\ [\mathcal{J}, X_i] & \text{if} \quad [\mathcal{J}, X_i] \neq 0. \end{cases} \tag{2.3}$$

(4) Enforce on the new generators X_i' the condition that they span a Lie algebra isomorphic to $\mathfrak{g}'$. If this is possible at all, this will determine the constants α_s.

Therefore, this four-step procedure firstly lies in the element $\mathcal{J}$ (2.2) which is formed by all the terms 'missing' in $U(\mathfrak{g})$ (according to the rank and the particular homogeneous space chosen for each particular deformation) in order to complete a base of invariants for $U(\mathfrak{g}')$. And secondly in the expression (2.3) which means that the new (deformed) generators directly come from such a 'missing' part in $U(\mathfrak{g})$, whilst the unchanged ones are those commuting with $\mathcal{J}$ remaining in the same form as in $U(\mathfrak{g})$.

We stress that the resulting expressions (2.3) may allow one to obtain operators and realizations of the deformed algebra $\mathfrak{g}'$ in terms of those of $U(\mathfrak{g})$ providing deformations for Lie algebra representations of the initial algebra.

It is worth noting a main difference of this approach with other proposals [7, 9, 13]: here not only the second-order $\mathcal{C}_1$ but *all* the Casimir invariants play a role in the deformation process, and the end result is intrinsic and would not change if a different choice for the Casimirs was used. In particular, we shall show that the fourth-order Pauli–Lubanski invariant is *essential* in order to recover the Poincaré algebra and some representations from the Galilei case. The need of using the higher-order Casimirs might explain why the inverse process to the Inönü–Wigner non-relativistic contraction has remained, as a matter of fact, as an unsolved problem in the framework of operators/representations. In this respect, we recall that it was already clear in the 1970s that higher-order Casimirs (and not only the quadratic one) would be involved in higher-rank deformations. This is explicitly stated by Gilmore [7], but apparently this line was not pursued enough.

3. The Galilei algebra and associated homogeneous spaces

In order to apply the above method to the $(3 + 1)$D Galilei algebra, $\mathcal{G} \equiv \mathfrak{iiso}(3)$, we first recall some basics. Let H, P_i, K_i and J_i be the usual generators of Galilean time translation, space translations, boosts and spatial rotations, respectively, with Lie brackets given by

$$\begin{array}{lll} [J_i, J_j] = \varepsilon_{ijk} J_k & [J_i, P_j] = \varepsilon_{ijk} P_k & [J_i, K_j] = \varepsilon_{ijk} K_k \\ [P_i, P_j] = 0 & [P_i, K_j] = 0 & [K_i, K_j] = 0 \\ [H, P_i] = 0 & [H, K_i] = -P_i & [H, J_i] = 0, \end{array} \tag{3.1}$$

where $i, j, k = 1, 2, 3$; sums over repeated indices will be generally assumed from now on. Hereafter we consider the $(3 + 1)$ notation and any 3-vector is denoted as $\mathbf{X} = (X_1, X_2, X_3)$; its square and its scalar and cross products with other vector, say $\mathbf{Y} = (Y_1, Y_2, Y_3)$, are

$$\begin{array}{ll} \mathbf{X}^2 = X_1^2 + X_2^2 + X_3^2 & |\mathbf{X}| = \left(X_1^2 + X_2^2 + X_3^2\right)^{1/2} \\ \mathbf{X} \cdot \mathbf{Y} = X_1 Y_1 + X_2 Y_2 + X_3 Y_3 & \mathbf{X} \wedge \mathbf{Y} = \mathbf{Z} \quad Z_i = \varepsilon_{ijk} X_j Y_k. \end{array} \tag{3.2}$$

Note that $\mathbf{X}$ and $\mathbf{Y}$ are generically non-commutative objects so that the ordering in these products must be preserved.

The Galilei algebra $\mathcal{G}$ has two Casimir invariants (quadratic and quartic) which read

$$\mathcal{C}_1 = \mathbf{P}^2 \qquad \mathcal{C}_2 = \mathbf{W}^2 \qquad \mathbf{W} = \mathbf{K} \wedge \mathbf{P} \tag{3.3}$$

corresponding to the square of the linear momentum $\mathbf{P}$ (non-relativistic kinetic energy) and to the square of the angular momentum $\mathbf{W}$, respectively.

Further computations within the universal enveloping Galilei algebra require use of the identities

$$\mathbf{P} \cdot \mathbf{W} = 0 \qquad \mathbf{K} \cdot \mathbf{W} = 0, \tag{3.4}$$

which imply that

$$|\mathbf{P} \wedge \mathbf{W}| = |\mathbf{P}||\mathbf{W}| \qquad |\mathbf{K} \wedge \mathbf{W}| = |\mathbf{K}||\mathbf{W}|. \tag{3.5}$$

We shall also need the following Lie brackets between the Galilei generators, the angular momentum components W_i and the products $\mathbf{J} \cdot \mathbf{P}$ and $\mathbf{J} \cdot \mathbf{W}$:

$$[W_i, H] = [W_i, P_j] = [W_i, K_j] = [W_i, W_j] = 0, \qquad [W_i, J_j] = \varepsilon_{ijk} W_k \tag{3.6}$$

$$\begin{aligned}
&[\mathbf{J} \cdot \mathbf{P}, H] = [\mathbf{J} \cdot \mathbf{P}, P_i] = [\mathbf{J} \cdot \mathbf{P}, J_i] = 0, \qquad [\mathbf{J} \cdot \mathbf{P}, K_i] = W_i \\
&[\mathbf{J} \cdot \mathbf{P}, W_i] = -\varepsilon_{ijk} P_j W_k, \qquad [\mathbf{J} \cdot \mathbf{W}, H] = [\mathbf{J} \cdot \mathbf{W}, J_i] = [\mathbf{J} \cdot \mathbf{W}, W_i] = 0 \\
&[\mathbf{J} \cdot \mathbf{W}, P_i] = \varepsilon_{ijk} P_j W_k, \qquad [\mathbf{J} \cdot \mathbf{W}, K_i] = \varepsilon_{ijk} K_j W_k, \qquad [\mathbf{J} \cdot \mathbf{W}, \mathbf{J} \cdot \mathbf{P}] = \varepsilon_{ijk} J_i P_j W_k.
\end{aligned} \tag{3.7}$$

Other commutators which involve the Casimir invariants are given by

$$\begin{aligned}
&[\mathbf{J} \cdot \mathbf{P}, \varepsilon_{ijk} P_j W_k] = \mathcal{C}_1 W_i, \qquad [\mathbf{J} \cdot \mathbf{P}, \varepsilon_{ijk} K_j W_k] = \mathbf{K} \cdot \mathbf{P} W_i \\
&[\mathbf{J} \cdot \mathbf{W}, \varepsilon_{ijk} P_j W_k] = -\mathcal{C}_2 P_i, \qquad [\mathbf{J} \cdot \mathbf{W}, \varepsilon_{ijk} K_j W_k] = -\mathcal{C}_2 K_i.
\end{aligned} \tag{3.8}$$

The involutive automorphisms parity P and the product PT [16] (where T is the time reversal)

$$\begin{aligned}
\mathsf{PT}: \quad & (H, \mathbf{P}, \mathbf{K}, \mathbf{J}) \to (-H, -\mathbf{P}, \mathbf{K}, \mathbf{J}) \\
\mathsf{P}: \quad & (H, \mathbf{P}, \mathbf{K}, \mathbf{J}) \to (H, -\mathbf{P}, -\mathbf{K}, \mathbf{J})
\end{aligned} \tag{3.9}$$

determine two Cartan decompositions of the Galilei Lie algebra $\mathcal{G}$,

$$\begin{aligned}
\mathsf{PT}: \quad & \mathcal{G} = \mathfrak{p}^{(1)} \oplus \mathfrak{h}^{(1)} \qquad \mathfrak{p}^{(1)} = \langle H, \mathbf{P} \rangle \qquad \mathfrak{h}^{(1)} = \langle \mathbf{K}, \mathbf{J} \rangle \\
\mathsf{P}: \quad & \mathcal{G} = \mathfrak{p}^{(2)} \oplus \mathfrak{h}^{(2)} \qquad \mathfrak{p}^{(2)} = \langle \mathbf{P}, \mathbf{K} \rangle \qquad \mathfrak{h}^{(2)} = \langle H, \mathbf{J} \rangle = \langle H \rangle \oplus \langle \mathbf{J} \rangle
\end{aligned} \tag{3.10}$$

fulfilling

$$[\mathfrak{h}^{(l)}, \mathfrak{h}^{(l)}] \subset \mathfrak{h}^{(l)} \qquad [\mathfrak{h}^{(l)}, \mathfrak{p}^{(l)}] \subset \mathfrak{p}^{(l)} \qquad [\mathfrak{p}^{(l)}, \mathfrak{p}^{(l)}] = 0 \qquad l = 1, 2. \tag{3.11}$$

Consequently, the Galilei group $G \equiv IISO(3)$ is the motion group of the two following symmetrical homogeneous spaces with dimension D_l, curvature κ_l and rank r_l:

$$\begin{aligned}
&\mathcal{S}^{(1)} = G/H^{(1)} = IISO(3)/ISO(3) \qquad\qquad D_1 = 4 \qquad \kappa_1 = 0 \qquad r_1 = 1 \\
&\mathcal{S}^{(2)} = G/H^{(2)} = IISO(3)/(\mathbb{R} \otimes SO(3)) \qquad D_2 = 6 \qquad \kappa_2 = 0 \qquad r_2 = 2.
\end{aligned}$$

The Galilei subgroups $H^{(1)}, H^{(2)}$ (whose Lie algebras are $\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}$) are the stabilizer subgroups of an event and of a time-like line, respectively. Therefore, the flat rank-1 space $\mathcal{S}^{(1)}$ is identified with the (3+1)D Galilean spacetime, while the flat rank-2 space $\mathcal{S}^{(2)}$ is the 6D space of time-like lines in the Galilean spacetime $\mathcal{S}^{(1)}$; we remark that $\mathcal{S}^{(2)}$ itself is the (classical) phase space of a free Galilean particle. Hence, this geometric approach makes two types of spaces to appear: spacetime on one side and phase spaces on the other. For a description of the structure of phase spaces behind the possible homogeneous spacetimes (Galilei, Newton–Hooke, Minkowski, (anti)de Sitter) as rank-2 spaces, see [24]. The phase spaces are the basic objects in the geometric quantization program, and through identification to orbits of the coadjoint action an interesting link to the study and classification of representations, the

'orbit method', has been developed by Souriau, Kostant and Kirillov (for a recent overview of this method, see [25]). Our aim here is more specific as we are not dealing with quantization at all.

Hence, as it is well known, two possible deformations arise from Galilei in a natural way. First, if we consider the Galilean spacetime $\mathcal{S}^{(1)}$ we can try to reach a kinematics whose associated spacetime has a *nonzero* curvature but keeps a *flat* phase space $\mathcal{S}^{(2)}$; that is, we keep non-relativistic (classical) mechanics. The resulting symmetries span the two Newton–Hooke (NH) algebras which are obtained through rank-1 *non-relativistic* deformations that provide *curved* spacetimes with curvature $\kappa_1 = \pm 1/\tau^2$ (where τ is the universe 'radius') but are still with *absolute time*. The opposite process is the spacetime contraction $\tau \to \infty$ in the NH algebras [16].

The second possibility is to start with the rank-2 space $\mathcal{S}^{(2)}$ and introducing a (negative) curvature in $\mathcal{S}^{(2)}$ as $\kappa_2 = -1/c^2$, while keeping a flat spacetime $\mathcal{S}^{(1)}$, and so reaching the Poincaré algebra. Hence, the usual (flat) *phase space* supporting classical mechanics becomes the proper 'curved' phase space underlying relativistic mechanics [24]. Alternatively, this process can be seen as a relativistic deformation which gives rise to the Minkowskian spacetime, still a *flat* universe (spacetime) but now without *absolute time*. This is exactly the opposite process to the well-known non-relativistic limit or speed-space contraction $c \to \infty$ [1, 2, 3, 16].

In the following two sections, we discuss in full explicit detail both types of Lie algebra deformations together with a study of the corresponding deformation of representations.

4. Poincaré from Galilei

In this section, we will see how the proposed deformation procedure leads to representations of the Poincaré algebra starting from those representations of the Galilei algebra with nonzero values for both Casimirs. These Galilei representations will be termed here as 'ordinary' just to emphasize that they are in some sense the 'generic ones'. At the representation level the procedure requires to restrict oneself to those 'ordinary' representations of the Galilei algebra to start with, and leads precisely to 'non-ordinary' representations of the Poincaré algebra, with both Casimirs vanishing.

The Lie brackets of the (3 + 1)D Poincaré algebra $\mathcal{P} \equiv \mathfrak{iso}(3, 1)$ read

$$[J_i', J_j'] = \varepsilon_{ijk} J_k' \qquad [J_i', P_j'] = \varepsilon_{ijk} P_k' \qquad [J_i', K_j'] = \varepsilon_{ijk} K_k'$$

$$[P_i', P_j'] = 0 \qquad [P_i', K_j'] = -\frac{1}{c^2}\delta_{ij} H' \qquad [K_i', K_j'] = -\frac{1}{c^2}\varepsilon_{ijk} J_k' \qquad (4.1)$$

$$[H', P_i'] = 0 \qquad [H', K_i'] = -P_i' \qquad [H', J_i'] = 0.$$

The two Poincaré Casimir invariants are given by

$$\mathcal{C}_1' = \mathbf{P}'^2 - \frac{1}{c^2} H'^2 \qquad \mathcal{C}_2' = \mathbf{W}'^2 - \frac{1}{c^2} W_0'^2$$

$$W_0' = \mathbf{J}' \cdot \mathbf{P}' \qquad W_i' = -\frac{1}{c^2} H' J_i' + \varepsilon_{ijk} K_j' P_k'. \qquad (4.2)$$

For an elementary system, the Casimir $\mathcal{C}_1'$ is, up to a factor, the square of the energy-momentum 4-vector $(H', \mathbf{P}')$; with the conventions here, this Casimir is equal to minus the system rest mass squared, while $\mathcal{C}_2'$ comes similarly from the square of the Pauli–Lubanski 4-vector $(W_0', \mathbf{W}')$, and is the square of the angular momentum in the rest frame; both quantities are the Poincaré invariants characterizing elementary systems.

The Cartan decompositions (3.10) also hold for the Poincaré algebra, though in this case the vector subspaces $\mathfrak{p}^{(l)}$ satisfy $[\mathfrak{p}^{(1)}, \mathfrak{p}^{(1)}] = 0$ and $[\mathfrak{p}^{(2)}, \mathfrak{p}^{(2)}] \subset \mathfrak{h}^{(2)}$ in the

relations (3.11). Therefore, the Poincaré group $P \equiv ISO(3, 1)$ is the motion group of the symmetric homogeneous spaces defined by

$$\mathcal{S}^{(1)} = P/H^{(1)} = ISO(3, 1)/SO(3, 1) \qquad\quad D_1 = 4 \qquad \kappa_1 = 0 \qquad r_1 = 1$$
$$\mathcal{S}^{(2)} = P/H^{(2)} = ISO(3, 1)/(\mathbb{R} \otimes SO(3)) \qquad D_2 = 6 \quad \kappa_2 = -1/c^2 \qquad r_2 = 2,$$

which are identified with the $(3 + 1)$D Minkowskian spacetime and the 6D (relativistic) phase space (or space of time-like lines), respectively.

The non-relativistic limit $c \to \infty$ leads to the contraction $\mathcal{P} \to \mathcal{G}$ and makes equal to zero the curvature of $\mathcal{S}^{(2)}$, so that the commutators $[P_i', K_j']$, $[K_i', K_j']$ are equal to zero in $\mathcal{G}$. Our aim now is to reverse this process, that is, to try to recover $\mathcal{P}$ inside the universal enveloping algebra of $\mathcal{G}$, $U(\mathcal{G})$, thus introducing the relativistic constant c.

4.1. The relativistic Galilei deformation

As the intended deformation $U(\mathcal{G}) \to \mathcal{P}$, which would produce the Poincaré algebra from the Galilei one, is a *rank-2 deformation*, the first step of the deformation method (2.1) requires us to rewrite *both* Poincaré Casimirs so as to explicitly display the powers of the curvature $-1/c^2$ of the phase space $\mathcal{S}^{(2)}$; this gives $\mathcal{J}_i^{(l)}$:

$$\mathcal{C}_1' = \mathcal{C}_1 - \frac{1}{c^2}\mathcal{J}_1^{(1)} \qquad \mathcal{J}_1^{(1)} = H^2$$
$$\mathcal{C}_2' = \mathcal{C}_2 - \frac{1}{c^2}\mathcal{J}_2^{(1)} + \left(-\frac{1}{c^2}\right)^2 \mathcal{J}_2^{(2)} \quad \mathcal{J}_2^{(1)} = 2H\mathbf{J}\cdot\mathbf{W} + (\mathbf{J}\cdot\mathbf{P})^2 \quad \mathcal{J}_2^{(2)} = H^2\mathbf{J}^2,$$

(4.3)

where $\mathcal{C}_1, \mathcal{C}_2$ are the *Galilei* Casimirs (3.3). We now consider the linear combination (2.2)

$$\mathcal{J} = \alpha_1\mathcal{J}_1^{(1)} + \alpha_2\mathcal{J}_2^{(1)} = \alpha_1 H^2 + \alpha_2(2H\mathbf{J}\cdot\mathbf{W} + (\mathbf{J}\cdot\mathbf{P})^2) \tag{4.4}$$

to be used as a seed to provide generators X_k' intended to close a Poincaré algebra. These are the elements of $U(\mathcal{G})$ obtained by following the third step (2.3), which requires commuting $\mathcal{J}$ with the Galilei generators X_k:

$$H' = H \qquad J_i' = J_i \qquad P_i' = 2\alpha_2 H\varepsilon_{ijk}P_j W_k$$
$$K_i' = -2\alpha_1 HP_i + 2\alpha_2\left(\mathbf{J}\cdot\mathbf{P}W_i - \mathbf{J}\cdot\mathbf{W}P_i + H\varepsilon_{ijk}K_j W_k + \tfrac{3}{2}\varepsilon_{ijk}P_j W_k\right).$$

(4.5)

The last step is to enforce the deformed generators (4.5) to fulfil the Lie brackets of the Poincaré algebra (4.1); the final result is established by

Proposition 1. *For a given irreducible representation of the Galilei algebra $\mathcal{G}$ with non-vanishing Casimirs $\mathcal{C}_1 \neq 0, \mathcal{C}_2 \neq 0$, the generators defined by (4.5) close the $(3 + 1)$D Poincaré algebra $\mathcal{P}$ whenever the constants α_1, α_2 are related to the Casimirs values by*

$$\alpha_1\mathcal{C}_1 + \alpha_2\mathcal{C}_2 = 0, \qquad \alpha_2^2 = \frac{1}{4c^2\mathcal{C}_1\mathcal{C}_2}. \tag{4.6}$$

Note that this process will only work for those irreducible representations of the starting Galilei algebra with both Casimirs *different from zero*. There is no standard naming for these representations, but as this condition is essential in our approach, we will refer here to these as 'ordinary' Galilei representations.

The proof of this result is a direct computation. Some generators stay unchanged in this deformation and span the subalgebra $\mathfrak{h}^{(2)} = \langle H, \mathbf{J} \rangle$. By taking into account the second Cartan decomposition (3.10) it can be checked that the assumptions of proposition 1 of [14] are automatically fulfilled, which implies that the Lie brackets $[\mathfrak{h}^{(2)}, \mathfrak{h}^{(2)}]$ and $[\mathfrak{h}^{(2)}, \mathfrak{p}^{(2)}]$ remain in

the same form as in the Galilei algebra, as it should be for Poincaré. Therefore, we have only to compute the commutators involving generators in $\mathfrak{p}^{(2)}$: $[P_i', P_j']$, $[P_i', K_j']$ and $[K_i', K_j']$.

As it is shown in (3.6), H, P_i and W_j commute amongst themselves, so that $[P_i', P_j'] = 0$. Let us compute $[P_i', K_l']$; by using (3.6) and (3.7) we obtain that

$$[P_i', K_l'] = -4\alpha_2^2 \varepsilon_{ijk} H[P_j, \mathbf{J} \cdot \mathbf{W}]P_l W_k + 4\alpha_2^2 \varepsilon_{ijk} H P_j[W_k, \mathbf{J} \cdot \mathbf{P}]W_l$$
$$+ 4\alpha_2^2 \varepsilon_{ijk} \varepsilon_{lmn} H[H, K_m]P_j W_k W_n$$
$$= 4\alpha_2^2 H \varepsilon_{ijk}(\varepsilon_{jmn} P_l W_k + \varepsilon_{kmn} P_j W_l - \varepsilon_{lmn} P_j W_k)P_m W_n.$$

Thus, if $i \neq l$, $[P_i', K_l'] = 0$, but whenever $i = l$ the Galilei Casimirs (3.3) can be recovered in the commutator as

$$[P_i', K_i'] = -4\alpha_2^2 H\mathbf{P}^2\mathbf{W}^2 = -4\alpha_2^2 H\mathcal{C}_1\mathcal{C}_2 = -4\alpha_2^2 \mathcal{C}_1\mathcal{C}_2 H'$$

and this will coincide with the expected Poincaré commutation relation whenever

$$-4\alpha_2^2\mathcal{C}_1\mathcal{C}_2 = -\frac{1}{c^2},$$

which leads to the second relation of (4.6). Next, let us compute

$$[K_i', K_l'] = 8\alpha_1\alpha_2 H(\varepsilon_{ljk} P_i - \varepsilon_{ijk} P_l)P_j W_k - 4\alpha_2^2 H\mathcal{C}_2\varepsilon_{ilk} W_k$$
$$- 4\alpha_2^2 H \varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{sjm} W_n W_s W_k - 4\alpha_2^2[\mathbf{J} \cdot \mathbf{W}, \mathbf{J} \cdot \mathbf{P}](P_i W_l - P_l W_i)$$
$$+ 4\alpha_2^2\{\mathbf{J} \cdot \mathbf{W}(\varepsilon_{ljk} P_i - \varepsilon_{ijk} P_l)P_j W_k + \mathbf{J} \cdot \mathbf{P}(\varepsilon_{ljk} W_i - \varepsilon_{ijk} W_l)W_j P_k\}.$$

By using identities (3.4), simplifying, grouping terms in order to construct the Galilei Casimirs and using the recently obtained relation we find that

$$[K_i', K_l'] = -8\alpha_2 H\varepsilon_{ilk}(\alpha_1\mathcal{C}_1 + \alpha_2\mathcal{C}_2)W_k - 4\alpha_2^2\varepsilon_{ilk}(\mathbf{J} \cdot \mathbf{W}\mathcal{C}_1 W_k + \mathbf{J} \cdot \mathbf{P}\mathcal{C}_2 P_k)$$
$$- 4\alpha_2^2\varepsilon_{mns} J_m P_n W_s(P_i W_l - P_l W_i)$$
$$= -8\alpha_2 H\varepsilon_{ilk}\{\alpha_1\mathcal{C}_1 + \alpha_2\mathcal{C}_2\}W_k - 4\alpha_2^2\varepsilon_{ilk} J_k\mathcal{C}_1\mathcal{C}_2$$
$$= -8\alpha_2 H\varepsilon_{ilk}\{\alpha_1\mathcal{C}_1 + \alpha_2\mathcal{C}_2\}W_k - \frac{1}{c^2}\varepsilon_{ilk} J_k',$$

so this coincides with the Poincaré commutator whenever

$$\alpha_1\mathcal{C}_1 + \alpha_2\mathcal{C}_2 = 0.$$

Hence proposition 1 is proven.

Consequently, this Galilean deformation reverses the non-relativistic contraction of the Poincaré algebra by *introducing* a negative curvature $\kappa_2 = -1/c^2$ in the 6D Galilean phase space $\mathcal{S}^{(2)}$. Recall that the space $\mathcal{S}^{(2)}$ is rank 2, so its geometry is *very* different from a rank-1 space, either curved or flat [24]. At the level of the Cartan decompositions the deformation gives $[\mathfrak{p}^{(2)}, \mathfrak{p}^{(2)}] = 0 \to [\mathfrak{p}^{(2)}, \mathfrak{p}^{(2)}] \subset \mathfrak{h}^{(2)}$.

4.2. Relativistic Poincaré operators through non-relativistic Galilei ones

Once the constants α_1 and α_2 have been determined, the expressions of the Poincaré generators in terms of those of the universal envolvent algebra $U(\mathcal{G})$ (4.5) of the Galilei algebra can be recast very compactly in vector form. We restrict from the beginning to 'ordinary' representations of the Galilei algebra, with $\mathcal{C}_1 \neq 0$ and $\mathcal{C}_2 \neq 0$. Choose the positive root determination for α_2, and use the notation (3.2), writing $\mathcal{C}_1^{1/2} = |\mathbf{P}|$ and $\mathcal{C}_2^{1/2} = |\mathbf{W}|$. For the deformed generators we obtain

$$c\mathbf{P}' = H\frac{\mathbf{P} \wedge \mathbf{W}}{|\mathbf{P}||\mathbf{W}|}, \qquad H' = H, \qquad \mathbf{J}' = \mathbf{J}$$

$$c\mathbf{K}' = H\left(\frac{|\mathbf{W}|}{\mathbf{P}^2}\frac{\mathbf{P}}{|\mathbf{P}|} + \frac{\mathbf{K}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|}\right) + \left(\frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}\frac{\mathbf{W}}{|\mathbf{W}|} - \frac{\mathbf{J}\cdot\mathbf{W}}{|\mathbf{W}|}\frac{\mathbf{P}}{|\mathbf{P}|}\right) + \frac{3}{2}\frac{\mathbf{P}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|}$$

$$= \frac{1}{|\mathbf{P}||\mathbf{W}|}\left\{H\left(\frac{\mathbf{W}^2}{\mathbf{P}^2}\mathbf{P} + \mathbf{K}\wedge\mathbf{W}\right) - \mathbf{J}\wedge(\mathbf{P}\wedge\mathbf{W}) + \frac{3}{2}\mathbf{P}\wedge\mathbf{W}\right\}. \tag{4.7}$$

Next we can deduce Poincaré operators as elements in $U(\mathcal{G})$. By using the relations (3.4)–(3.8) we find that

$$c^2\mathbf{P}'^2 = H'^2 \qquad c^2\mathbf{K}'\wedge\mathbf{P}' = H\mathbf{J} - H\left\{\frac{\mathbf{J}\cdot(\mathbf{P}\wedge\mathbf{W})}{\mathbf{P}^2\mathbf{W}^2}\right\}\mathbf{P}\wedge\mathbf{W}. \tag{4.8}$$

Thus the Pauli–Lubanski components turn out to be

$$W_0' = \frac{1}{c}H\mathbf{J}\cdot\frac{(\mathbf{P}\wedge\mathbf{W})}{|\mathbf{P}||\mathbf{W}|} \qquad \mathbf{W}' = -\frac{1}{c^2}H\left\{\frac{\mathbf{J}\cdot(\mathbf{P}\wedge\mathbf{W})}{|\mathbf{P}||\mathbf{W}|}\right\}\frac{\mathbf{P}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|}. \tag{4.9}$$

These expressions explicitly show how the component W_0', not appearing in the Galilei case (3.3), can be constructed and, moreover, they lead to

$$\mathbf{W}' = -\frac{1}{c}W_0'\frac{\mathbf{P}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|} \qquad c^2\mathbf{W}'^2 = W_0'^2. \tag{4.10}$$

If we now substitute these expressions into the Poincaré Casimirs $\mathcal{C}_1', \mathcal{C}_2'$ given in (4.2) it can be checked that both of them vanish. This, in turn, means that under this deformation *any irreducible Galilean representation with both $\mathcal{C}_1 \neq 0, \mathcal{C}_2 \neq 0$ gives rise to a *massless* and spinless* Poincaré representation with both $\mathcal{C}_1' = \mathcal{C}_2' = 0$. Hence, the expressions (4.7) can be interpreted as a correspondence between non-relativistic 'ordinary' systems and relativistic zero-mass systems. This suggests some kind of relationship between these operators and the null-plane (or light-cone) Poincaré framework [26].

Furthermore, a remarkable fact is the appearance in (4.7) of a Galilean angular momentum [27]

$$\lambda_p = \frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|} = \mathbf{J}\cdot\mathbf{u}_p, \tag{4.11}$$

which generates an $\mathfrak{so}(2)$ Lie algebra and corresponds to the projection of $\mathbf{J}$ along the direction $\mathbf{u}_p$ of the motion $\mathbf{P}$; recall that we have required $\mathcal{C}_1^{1/2} = |\mathbf{P}| \neq 0$. Similarly, we can consider two other helicity-type operators defined along two orthogonal directions to $\mathbf{P}$ determined by $\mathbf{W} = \mathbf{K}\wedge\mathbf{P}$ and $\mathbf{P}\wedge\mathbf{W}$ (note also that accordingly $\mathcal{C}_2^{1/2} = |\mathbf{W}|$ has been assumed to be different from zero):

$$\lambda_w := \frac{\mathbf{J}\cdot\mathbf{W}}{|\mathbf{W}|} = \mathbf{J}\cdot\mathbf{u}_w \qquad \lambda_{pw} := \frac{\mathbf{J}\cdot(\mathbf{P}\wedge\mathbf{W})}{|\mathbf{P}\wedge\mathbf{W}|} = \mathbf{J}\cdot\mathbf{u}_{pw}. \tag{4.12}$$

By taking into account (3.7) and (3.8), we find that these three Galilean 'angular momenta' close an $\mathfrak{so}(3)$ Lie algebra:

$$[\lambda_w, \lambda_p] = \lambda_{pw} \qquad [\lambda_{pw}, \lambda_w] = \lambda_p \qquad [\lambda_p, \lambda_{pw}] = \lambda_w. \tag{4.13}$$

In this way, we can write the Poincaré angular momentum λ' and the Pauli–Lubanski components in terms of the Galilean 'helicity' λ_{pw} in the form

$$\lambda' = c\frac{W_0'}{H'} = \frac{\mathbf{J}'\cdot\mathbf{P}'}{|\mathbf{P}'|} \equiv \lambda_{pw} \qquad W_0' = \frac{1}{c}H\lambda_{pw} \qquad \mathbf{W}' = -\frac{1}{c^2}H\lambda_{pw}\mathbf{u}_{pw}. \tag{4.14}$$

Now we consider the Poincaré operators Q_i' defined by Bacry [28] (of course, for $H' \neq 0$),

$$Q_i' = \frac{c^2}{2}\left(\frac{1}{H'}K_i' + K_i'\frac{1}{H'}\right). \tag{4.15}$$

By introducing (4.7) and applying the relations (3.4)–(3.8) we find that these operators are expressed through elements of $U(\mathcal{G})$ as

$$
\frac{1}{c}\mathbf{Q}' = \frac{|\mathbf{W}|}{\mathbf{P}^2}\frac{\mathbf{P}}{|\mathbf{P}|} + \frac{\mathbf{K}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|} + \frac{1}{H}\left(\frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}\frac{\mathbf{W}}{|\mathbf{W}|} - \frac{\mathbf{J}\cdot\mathbf{W}}{|\mathbf{W}|}\frac{\mathbf{P}}{|\mathbf{P}|} + \frac{\mathbf{P}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|}\right)
$$

$$
= \frac{|\mathbf{W}|}{\mathbf{P}^2}\mathbf{u}_p + \frac{\mathbf{K}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|} + \frac{1}{H}(\lambda_p\mathbf{u}_w - \lambda_w\mathbf{u}_p + \mathbf{u}_{pw}). \tag{4.16}
$$

Some expected properties fulfilled by $\mathbf{Q}'$ are consistently recovered under deformation. Working within $U(\mathcal{G})$ we obtain

$$
[J_i', Q_j'] = \varepsilon_{ijk}Q_k' \qquad [Q_i', P_j'] = \delta_{ij}. \tag{4.17}
$$

The former Lie bracket shows that $\mathbf{Q}'$ behaves as a 3-vector under rotations $\mathbf{J}' = \mathbf{J}$, while the latter provides the natural canonical commutation relations between Galilean position and momenta operators; had we kept a definition for Galilean position operators Q_i analogous to (4.15) then we would had obtained the unwanted result $[Q_i, P_j] = 0$. Secondly, cumbersome computations lead to

$$
[Q_i', Q_j'] = \frac{c^2}{H^2}\varepsilon_{ijk}\left(\frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}\frac{P_k}{|\mathbf{P}|} + \frac{\mathbf{J}\cdot\mathbf{W}}{|\mathbf{W}|}\frac{W_k}{|\mathbf{W}|} - J_k\right)
$$

$$
= \frac{c^2}{H'^2}(Q_i'P_j' - Q_j'P_i' - \varepsilon_{ijk}J_k'). \tag{4.18}
$$

As a byproduct, the Poincaré kinematical observables $\Sigma_k' = J_k' - \varepsilon_{ijk}Q_i'P_j'$ can be written in terms of Galilean operators as

$$
\Sigma' = \mathbf{J} - \frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}\frac{\mathbf{P}}{|\mathbf{P}|} - \frac{\mathbf{J}\cdot\mathbf{W}}{|\mathbf{W}|}\frac{\mathbf{W}}{|\mathbf{W}|} = \mathbf{J} - \lambda_p\mathbf{u}_p - \lambda_w\mathbf{u}_w. \tag{4.19}
$$

Finally, the Poincaré velocity operator $\mathbf{V}'$ turns out to be

$$
\mathbf{V}' = [\mathbf{Q}', H'] = c\frac{\mathbf{P}\wedge\mathbf{W}}{|\mathbf{P}||\mathbf{W}|} = c^2\frac{\mathbf{P}'}{H'} = c\frac{\mathbf{P}'}{|\mathbf{P}'|}, \tag{4.20}
$$

provided that $H' = c|\mathbf{P}'|$ since this deformation gives rise to Poincaré massless representations. The result (4.20) can be interpreted as a relationship between the motion of a non-relativistic particle along the direction $\mathbf{u}_{pw}$ with speed c and the motion of a free massless relativistic particle along the direction $\mathbf{u}_{p'}'$.

5. Newton–Hooke from centrally extended Galilei

Besides Poincaré, the Galilei algebra has two other kinematical neighbours: the oscillating $\mathcal{N}_+$ and the expanding $\mathcal{N}_-$ NH algebras [16]. These are the Lie algebras of the motion groups of *absolute time* universes with *nonzero curvature* $\kappa_1 = \pm 1/\tau^2$, where τ is the universe time radius. The commutation rules of $\mathcal{N}_\pm$ are given by

$$
\begin{array}{lll}
[J_i', J_j'] = \varepsilon_{ijk}J_k' & [J_i', P_j'] = \varepsilon_{ijk}P_k' & [J_i', K_j'] = \varepsilon_{ijk}K_k' \\
[P_i', P_j'] = 0 & [P_i', K_j'] = 0 & [K_i', K_j'] = 0 \\
[H', P_i'] = \kappa_1 K_i' & [H', K_i'] = -P_i' & [H', J_i'] = 0.
\end{array} \tag{5.1}
$$

The Casimirs of $\mathcal{N}_\pm$ turn out to be

$$
\mathcal{C}_1' = \mathbf{P}'^2 + \kappa_1\mathbf{K}'^2 \qquad \mathcal{C}_2' = \mathbf{W}'^2 \tag{5.2}
$$

where the components of $\mathbf{W}'$ are formally identical to the Galilean ones (3.3).

Due to the nonzero Lie brackets $[H', P_i'] = \kappa_1 K_i'$, the vector subspaces $\mathfrak{p}^{(l)}$ of the Cartan decompositions (3.10) now verify $[\mathfrak{p}^{(1)}, \mathfrak{p}^{(1)}] \subset \mathfrak{h}^{(1)}$ and $[\mathfrak{p}^{(2)}, \mathfrak{p}^{(2)}] = 0$. The NH groups $N_\pm$ are the motion groups of the following symmetric homogeneous spaces

$$
\begin{aligned}
\mathcal{S}^{(1)} &= N_+/H^{(1)} = T_6(SO(2) \otimes SO(3))/ISO(3) & \kappa_1 &= 1/\tau^2 \\
\mathcal{S}^{(2)} &= N_+/H^{(2)} = T_6(SO(2) \otimes SO(3))/(SO(2) \otimes SO(3)) & \kappa_2 &= 0 \\
\mathcal{S}^{(1)} &= N_-/H^{(1)} = T_6(SO(1,1) \otimes SO(3))/ISO(3) & \kappa_1 &= -1/\tau^2 \\
\mathcal{S}^{(2)} &= N_-/H^{(2)} = T_6(SO(1,1) \otimes SO(3))/(SO(1,1) \otimes SO(3)) & \kappa_2 &= 0.
\end{aligned}
$$

The spaces $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ correspond, in this order, to the $(3+1)$D non-relativistic curved spacetime [24, 29] and the 6D flat phase space (or space of time-like worldlines in the spacetime).

The limit $\kappa_1 \to 0$ ($\tau \to \infty$) produces the spacetime contraction $\mathcal{N}_\pm \to \mathcal{G}$. We consider the opposite process which would introduce a universe time radius τ in $U(\mathcal{G})$. As we are dealing with rank-1 deformations we only consider the first Casimir (5.2), so that the 'seed' element (2.2) is simply $\mathcal{J} = \alpha_1 \mathcal{J}_1^{(1)} = \alpha_1 \mathbf{K}^2$. The deformed generators (2.3) are $H' = 2\alpha_1 \mathbf{K} \cdot \mathbf{P}$ and all the remaining ones unchanged. However these new generators *do not span* $\mathcal{N}_\pm$. Hence, it is necessary to take an initial Lie algebra *less Abelian* in order to be able to perform the deformation. Thus we start from a central extension of $\mathcal{G}$ (as in lower dimensions [14, 30]), with central generator Ξ and central charge m (the mass of the Galilean particle as a Galilean elementary system [15]). The commutation rules of the centrally extended Galilei algebra $\overline{\mathcal{G}}$ are given by (3.1) once the vanishing brackets $[K_i, P_j]$ are replaced by

$$
[K_i, P_j] = \delta_{ij} m \Xi \qquad [\Xi, \cdot] = 0. \tag{5.3}
$$

We now apply (2.3) with $\mathcal{J} = \alpha_1 \mathbf{K}^2$ for the Lie brackets of $\overline{\mathcal{G}}$, finding the deformed generators, which live on the universal enveloping algebra $U(\overline{\mathcal{G}})$:

$$
J_i' = J_i \qquad K_i' = K_i \qquad H' = 2\alpha_1 \mathbf{K} \cdot \mathbf{P} - 3\alpha_1 m \Xi \qquad P_i' = 2\alpha_1 m \Xi K_i. \tag{5.4}
$$

Then this deformation is characterized by

Proposition 2. *The generators defined in terms of the (extended, with $m \neq 0$) Galilean ones by (5.4) span the $(3+1)$D NH algebras provided that the constant α_1 fulfils*

$$
\alpha_1^2 = -\frac{\kappa_1}{4m^2 \Xi^2} = \mp \frac{1}{4\tau^2 m^2 \Xi^2} \qquad for \quad \mathcal{N}_\pm. \tag{5.5}
$$

Proof. The generators which are unchanged in the deformation close the isotopy subalgebra of an event $\mathfrak{h}^{(1)} = \langle \mathbf{K}, \mathbf{J} \rangle$. Thus the proposition 1 of [14] can be applied and we only need to compute the Lie brackets involving generators of $\mathfrak{p}^{(1)}$. By direct computations we obtain the relation (5.5),

$$
[P_i', P_j'] = 0 \qquad [H', P_i'] = -4\alpha_1^2 m^2 \Xi^2 K_i \equiv \kappa_1 K_i'.
$$

Therefore, this deformation introduces a constant curvature κ_1 in the flat spacetime $G/H^{(1)}$, leading to curved spacetimes $N_\pm/H^{(1)}$ with the transition $[\mathfrak{p}^{(1)}, \mathfrak{p}^{(1)}] = 0 \to [\mathfrak{p}^{(1)}, \mathfrak{p}^{(1)}] \subset \mathfrak{h}^{(1)}$. All the phase spaces $\mathcal{S}^{(2)}$ remain flat. $\qquad \square$

5.1. Deformation of Galilei representations

If we set $\alpha_1 = \sqrt{-\kappa_1}/(2m\Xi)$ and substitute in the deformed generators (5.4), we find that these read

$$
H' = \sqrt{-\kappa_1} \left(\frac{\mathbf{K} \cdot \mathbf{P}}{m \Xi} - \frac{3}{2} \right) \qquad \mathbf{P}' = \sqrt{-\kappa_1} \mathbf{K}. \tag{5.6}
$$

Hence $\mathcal{C}'_1 = 0$ and $\mathbf{W}' = \mathbf{K}' \wedge \mathbf{P}' = 0$. Consequently, any irreducible representation of $\overline{\mathcal{G}}$ with $m \neq 0$ gives rise to a representation of $\mathcal{N}_\pm$ with zero Casimirs eigenvalues as both deformed $\mathcal{C}'_l$ (5.2) vanish. In particular, let us consider the representation of $\overline{\mathcal{G}}$ in terms of the momenta $\mathbf{p} = (p_1, p_2, p_3)$ given by Bacry [27]:

$$J_i = \varepsilon_{ijk} p_k \frac{\partial}{\partial p_j} + S_i \qquad K_i = m \frac{\partial}{\partial p_i} \qquad H = \frac{\mathbf{p}^2}{2m} + a, \qquad P_i = p_i \qquad \Xi = 1, \quad (5.7)$$

where a is a constant and S_i are spin operators fulfilling $[S_i, S_j] = \varepsilon_{ijk} S_k$. Then the relations (5.6) lead to the following representation for the transformed generators of $\mathcal{N}_\pm$:

$$H' = \sqrt{-\kappa_1} \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}} + \frac{3}{2} \right) \qquad \mathbf{P}' = \sqrt{-\kappa_1} m \frac{\partial}{\partial \mathbf{p}}, \qquad (5.8)$$

keeping $\mathbf{J}' = \mathbf{J}$ and $\mathbf{K}' = \mathbf{K}$.

6. (Anti)de Sitter from Poincaré

Known Lie algebra deformations $\mathfrak{iso}(p, q) \rightarrow \mathfrak{so}(p + 1, q)$ [7, 9] can be recovered from our approach, which also provides the explicit form of the operators representing the deformed generators inside the initial universal enveloping algebra. As an example, in this section, we give either anti-de Sitter $\mathcal{S}_+ = \mathfrak{so}(3, 2)$ or de Sitter $\mathcal{S}_- = \mathfrak{so}(4, 1)$ algebras and representations as deformations of Poincaré ones. To maintain consistency with the notation in the previous part of the paper, we will denote here X' any element of $U(\mathcal{P})$ (as (4.1) and (4.2)) and X'' any (anti)de Sitter object. The (anti)de Sitter commutation relations read

$$
\begin{array}{lll}
[J''_i, J''_j] = \varepsilon_{ijk} J''_k & [J''_i, P''_j] = \varepsilon_{ijk} P''_k & [J''_i, K''_j] = \varepsilon_{ijk} K''_k \\
[P''_i, P''_j] = -\frac{\kappa_1}{c^2} \varepsilon_{ijk} J''_k & [P''_i, K''_j] = -\frac{1}{c^2} \delta_{ij} H'' & [K''_i, K''_j] = -\frac{1}{c^2} \varepsilon_{ijk} J''_k \\
[H'', P''_i] = \kappa_1 K''_i & [H'', K''_i] = -P''_i & [H'', J''_i] = 0,
\end{array}
\qquad (6.1)
$$

where κ_1 is the curvature of the spacetime: $\kappa_1 = 1/\tau^2$ for $\mathcal{S}_+$ and $\kappa_1 = -1/\tau^2$ for $\mathcal{S}_-$. In both cases, the constant τ plays the role of the curvature radius of the (anti)de Sitter universe; we recall that in our conventions τ is dimensionally a time (so the 'length radius' would be $c\tau$) and the sign of spacetime curvature has been chosen so that κ_1 is related to the acceleration of separation of *time-like* geodesics with the same sign as in the Riemannian case (the sign is opposite for space-like geodesics). The Casimir invariants are [22]

$$
\begin{aligned}
\mathcal{C}''_1 &= \mathbf{P}''^2 - \frac{1}{c^2} H''^2 + \kappa_1 \left(\mathbf{K}''^2 - \frac{1}{c^2} \mathbf{J}''^2 \right) \\
\mathcal{C}''_2 &= \mathbf{W}''^2 - \frac{1}{c^2} W''^2_0 - \frac{\kappa_1}{c^2} (\mathbf{J}'' \cdot \mathbf{K}'')^2,
\end{aligned}
\qquad (6.2)
$$

where $\mathbf{W}''$, W''_0 are formally identical to the Poincaré Pauli–Lubanski components (4.2). Now, both vector subspaces $\mathfrak{p}^{(l)}$ (3.10) fulfil $[\mathfrak{p}^{(l)}, \mathfrak{p}^{(l)}] \subset \mathfrak{h}^{(l)}$, and the symmetric homogeneous (anti)de Sitter spaces are defined by

$$
\begin{array}{ll}
\mathcal{S}^{(1)} = AdS/H^{(1)} = SO(3, 2)/SO(3, 1) & \kappa_1 = 1/\tau^2 \\
\mathcal{S}^{(2)} = AdS/H^{(2)} = SO(3, 2)/(SO(2) \otimes SO(3)) & \kappa_2 = -1/c^2 \\
\mathcal{S}^{(1)} = dS/H^{(1)} = SO(4, 1)/SO(3, 1) & \kappa_1 = -1/\tau^2 \\
\mathcal{S}^{(2)} = dS/H^{(2)} = SO(4, 1)/(SO(1, 1) \otimes SO(3)) & \kappa_2 = -1/c^2.
\end{array}
$$

Hence, the spaces $\mathcal{S}^{(1)}$ are the curved relativistic spacetimes (with the two possible signs of the curvature), and the two 'phase spaces' $\mathcal{S}^{(2)}$ are the corresponding spaces of time-like lines; both are (negatively) curved phase spaces.

The deformation $U(\mathcal{P}) \rightarrow \{\mathcal{S}_\pm\}$ is rank 1 with underlying space $\mathcal{S}^{(1)}$ and curvature κ_1. Then only the first Casimir $\mathcal{C}_1''$ (6.2) determines the 'seed' element (2.2),

$$\mathcal{J} = \alpha_1 \left(\mathbf{K}''^2 - \frac{1}{c^2} \mathbf{J}''^2 \right), \tag{6.3}$$

so that the deformed generators, defined on $U(\mathcal{P})$, turn out to be

$$J_i'' = J_i' \qquad K_i'' = K_i' \qquad H'' = \frac{2\alpha_1}{c^2} \left(c^2 \mathbf{K}' \cdot \mathbf{P}' - \frac{3}{2} H' \right)$$

$$P_i'' = \frac{2\alpha_1}{c^2} \left(K_i' H' + \varepsilon_{ijk} J_j' P_k' - \frac{3}{2} P_i' \right). \tag{6.4}$$

The constant α_1 is determined similarly to the NH case.

Proposition 3. *For a given irreducible representation of the Poincaré algebra with second-order Poincaré Casimir $\mathcal{C}_1' \neq 0$, the generators defined by (6.4) close the $(3+1)D$ (anti)de Sitter algebras whenever the constant α_1 is related to the value $\mathcal{C}_1'$ by*

$$\alpha_1^2 = \frac{\kappa_1 c^2}{4\mathcal{C}_1'} = \pm \frac{c^2}{4\tau^2 \mathcal{C}_1'}. \tag{6.5}$$

Note that within our approach light-like representations of the Poincaré algebra cannot provide deformations to (anti)de Sitter representations.

Proof. As in the previous section the isotopy subalgebra $\mathfrak{h}^{(1)}$ of an event remains invariant under this deformation, thus we only compute the Lie brackets for the translation generators that belong to $\mathfrak{p}^{(1)}$; these directly give (6.5),

$$[P_i'', P_j''] = -\frac{4\alpha_1^2}{c^4} \varepsilon_{ijk} \mathcal{C}_1' J_k' \equiv -\frac{\kappa_1}{c^2} \varepsilon_{ijk} J_k'' \qquad [H'', P_i''] = \frac{4\alpha_1^2}{c^2} \mathcal{C}_1' K_i' \equiv \kappa_1 K_i''. \qquad \square$$

6.1. Deformation of Poincaré representations

Further to the above algebraic results, which are in agreement with those established in [7, 9], we explore the deformation of representations.

By taking into account the deformed generators (6.4) with the solution (6.5), it can be proven that the (anti)de Sitter invariants (6.2) are expressed in terms of the Poincaré Casimirs (4.2) as

$$\mathcal{C}_1'' = -\frac{9\kappa_1}{4c^2} + \kappa_1 \frac{\mathcal{C}_2'}{\mathcal{C}_1'} \qquad \mathcal{C}_2'' = -\frac{\kappa_1}{4c^2} \frac{\mathcal{C}_2'}{\mathcal{C}_1'}. \tag{6.6}$$

Since this Poincaré deformation is only possible when $\mathcal{C}_1' \neq 0$, the result (6.6) means that, in general, any massive Poincaré particle gives rise generically to an (anti)de Sitter representation with $\mathcal{C}_1'' \neq 0$, with a singular behaviour when $\mathcal{C}_2' = 9\mathcal{C}_1'/(4c^2)$ which gives $\mathcal{C}_1'' = 0$. As far as the second Casimir it is clear that $\mathcal{C}_2' = 0$ leads to $\mathcal{C}_2'' = 0$.

In particular, let us consider the irreducible Poincaré representation of mass $m \neq 0$ and spin $s \neq 0$ in the momentum **p**-space [27],

$$H' = c\sqrt{\mathbf{p}^2 + m^2 c^2} \qquad P_i' = p_i \qquad J_i' = \varepsilon_{ijk} p_k \frac{\partial}{\partial p_j} + S_i$$

$$K_i' = \frac{1}{c} \left(\sqrt{\mathbf{p}^2 + m^2 c^2} \frac{\partial}{\partial p_i} + \frac{\varepsilon_{ijk} p_j S_k}{mc + \sqrt{\mathbf{p}^2 + m^2 c^2}} \right), \tag{6.7}$$

13

where S_i are spin operators such that the eigenvalue of $\mathbf{S}^2$ is $s(s+1)$; note that $p^0 = c\sqrt{\mathbf{p}^2 + m^2c^2}$. This representation is characterized by the eigenvalues of both Casimirs $\mathcal{C}'_l$ (4.2),

$$\mathcal{C}'_1 = -m^2c^2 \qquad \mathcal{C}'_2 = m^2\mathbf{S}^2. \tag{6.8}$$

Then we find that the eigenvalues of the (anti)de Sitter Casimirs (6.6) read

$$\mathcal{C}''_1 = -\frac{\kappa_1}{c^2}\left(\frac{9}{4} + \mathbf{S}^2\right) \qquad \mathcal{C}''_2 = \frac{\kappa_1}{4c^4}\mathbf{S}^2, \tag{6.9}$$

so that, in this case $\mathcal{C}''_1 \neq 0$ for any non-vanishing spin. Finally, by taking $\alpha_1 = \sqrt{-\kappa_1}/(2m)$, the deformed generators (6.4) yield to the following (anti)de Sitter representation:

$$
\begin{aligned}
H'' &= \frac{\sqrt{-\kappa_1}}{mc}\sqrt{\mathbf{p}^2 + m^2c^2}\left(\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}} + \frac{3}{2}\right) \\
P''_i &= \sqrt{-\kappa_1}\left(\frac{p_i}{mc^2}\left(\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}} + \frac{3}{2}\right) + m\frac{\partial}{\partial p_i} - \frac{\varepsilon_{ijk}p_jS_k}{mc^2 + c\sqrt{\mathbf{p}^2 + m^2c^2}}\right),
\end{aligned} \tag{6.10}
$$

with $\mathbf{J}'' = \mathbf{J}'$ and $\mathbf{K}'' = \mathbf{K}'$.

7. Outlook: beyond classical deformations

The deformation method proposed in [14] for the $(2+1)$D kinematical algebras has been generalized in order to be applicable to higher dimensions and to any Lie algebra, and has been explicitly applied to the $(3+1)$D Galilei algebra. The method works in the two physically meaningful 'deformation directions' which lead either to the Poincaré algebra or to the NH ones. Deformations going from Poincaré to both de Sitter algebras have been recovered as well. Furthermore, beyond the pure algebraic constructions, we have presented a preliminary study of deformations of operators and representations which may deserve deeper development.

For the deformation from Galilei to Poincaré, the method as described here does not provide all the Poincaré representations, but only the 'light-like' ones associated with the vanishing of both Poincaré Casimirs. To extend the method so as to allow the general representations of the Poincaré algebra remains as an open problem.

Clearly, the same procedure also holds for the rank-2 deformation going from Galilei to the Euclidean algebra $\mathcal{E} \equiv \mathfrak{iso}(4)$. If we replace in *all* the Poincaré expressions given in section 4, the negative curvature $\kappa_2 = -1/c^2$ by a positive one (set c equal to a pure imaginary complex number), then we would obtain all the Euclidean structures. The deformation $U(\mathcal{G}) \to \mathcal{E}$ would lead to similar deformed generators and conditions as those characterized by proposition 1. In this case, the symmetric space $\mathcal{S}^{(1)} = ISO(4)/SO(4)$ is the 4D flat Euclidean space and $\mathcal{S}^{(2)} = ISO(4)/(\mathbb{R} \otimes SO(3))$ is a positively curved and rank-2 6D space of lines in the Euclidean space. Deformation of Galilean representations are characterized by the deformed Euclidean Casimirs $\mathcal{C}'_l$ whose eigenvalues are again equal to zero. Summing up, we have obtained the following rank-2 Galilei deformations:

4D Euclidean		(3+1) D Galilei		(3 + 1)D Poincaré
$\mathcal{E} \equiv \mathfrak{iso}(4)$	$\longleftarrow$	$U(\mathcal{G}) \equiv U(\mathfrak{iiso}(3))$	$\longrightarrow$	$\mathcal{P} \equiv \mathfrak{iso}(3,1)$
$\kappa_1 = 0, \kappa_2 > 0$		$\kappa_1 = 0, \kappa_2 = 0(c = \infty)$		$\kappa_1 = 0, \kappa_2 = -1/c^2$
$\mathcal{C}'_1 = 0, \mathcal{C}'_2 = 0$		$\mathcal{C}_1 \neq 0, \mathcal{C}_2 \neq 0$		$\mathcal{C}'_1 = 0, \mathcal{C}'_2 = 0$

In the same way, similar rank-1 deformations to those described in section 6 but now starting from $U(\mathcal{E})$ instead of $U(\mathcal{P})$ can also be obtained. They would give rise to either $\mathfrak{so}(5)(\kappa_1 > 0)$ or $\mathfrak{so}(4,1)(\kappa_1 < 0)$, with underlying homogeneous spaces $\mathcal{S}^{(l)}$ given in this

order by (i) the 4D sphere $\mathcal{S}^{(1)} = SO(5)/SO(4)$ with radius R and curvature $\kappa_1 = 1/R^2$, and the Grassmannian $\mathcal{S}^{(2)} = SO(5)/(SO(2) \otimes SO(3))$ with $\kappa_2 > 0$; (ii) the 4D hyperbolic space $\mathcal{S}^{(1)} = SO(4, 1)/SO(4)$, with $\kappa_1 = -1/R^2$ and the corresponding Grassmannian $\mathcal{S}^{(2)} = SO(4, 1)/(SO(1, 1) \otimes SO(3))$ with $\kappa_2 > 0$. In both processes the eigenvalue of the Euclidean Casimir $\mathcal{C}_1'$ must be different from zero and deformation of Euclidean representations are determined by the deformed Casimirs $\mathcal{C}_l''$ (6.6) with c purely imaginary (say $c = \mathrm{i}$, so that $\kappa_2 = +1$). Hence, the rank-1 deformations studied here (covering sections 5 and 6) are summarized as

4D Sphere		4D Euclidean		4D Hyperbolic
$\mathfrak{so}(5)$	$\longleftarrow$	$U(\mathcal{E}) \equiv U(\mathfrak{iso}(4))$	$\longrightarrow$	$\mathfrak{so}(4, 1)$
$\kappa_1 = +1/R^2, \kappa_2 = +1$		$\kappa_1 = 0, \kappa_2 = +1$		$\kappa_1 = -1/R^2, \kappa_2 = +1$
$\mathcal{C}_1'' = \frac{1}{R^2}\left(\frac{9}{4} + \frac{\mathcal{C}_2'}{\mathcal{C}_1'}\right)$		$\mathcal{C}_1' \neq 0, \mathcal{C}_2'$ arbitrary		$\mathcal{C}_1'' = -\frac{1}{R^2}\left(\frac{9}{4} + \frac{\mathcal{C}_2'}{\mathcal{C}_1'}\right)$
$\mathcal{C}_2'' = \frac{1}{4R^2}\frac{\mathcal{C}_2'}{\mathcal{C}_1'}$				$\mathcal{C}_2'' = -\frac{1}{4R^2}\frac{\mathcal{C}_2'}{\mathcal{C}_1'}$
(3 + 1)D Anti-de Sitter		(3 + 1)D Poincaré		(3+1)D de Sitter
$\mathcal{S}_+ \equiv \mathfrak{so}(3, 2)$	$\longleftarrow$	$U(\mathcal{P}) \equiv U(\mathfrak{iso}(3, 1))$	$\longrightarrow$	$\mathcal{S}_- \equiv \mathfrak{so}(4, 1)$
$\kappa_1 = +1/\tau^2, \kappa_2 = -1/c^2$		$\kappa_1 = 0, \kappa_2 = -1/c^2$		$\kappa_1 = -1/\tau^2, \kappa_2 = -1/c^2$
$\mathcal{C}_1'' = -\frac{1}{\tau^2}\left(\frac{9}{4c^2} - \frac{\mathcal{C}_2'}{\mathcal{C}_1'}\right)$		$\mathcal{C}_1' \neq 0, \mathcal{C}_2'$ arbitrary		$\mathcal{C}_1'' = \frac{1}{\tau^2}\left(\frac{9}{4c^2} - \frac{\mathcal{C}_2'}{\mathcal{C}_1'}\right)$
$\mathcal{C}_2'' = -\frac{1}{4\tau^2 c^2}\frac{\mathcal{C}_2'}{\mathcal{C}_1'}$				$\mathcal{C}_2'' = \frac{1}{4\tau^2 c^2}\frac{\mathcal{C}_2'}{\mathcal{C}_1'}$
(3+1)D Oscillating NH		(3+1)D Extended Galilei		(3+1)D Expanding NH
$\mathcal{N}_+ \equiv \mathfrak{t}_6(\mathfrak{so}(2) \oplus \mathfrak{so}(3))$	$\longleftarrow$	$U(\overline{\mathcal{G}}) \equiv U(\overline{\mathfrak{iiso}(3)})$	$\longrightarrow$	$\mathcal{N}_- \equiv \mathfrak{t}_6(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(3))$
$\kappa_1 = +1/\tau^2, \kappa_2 = 0$		$\kappa_1 = 0, \kappa_2 = 0$		$\kappa_1 = -1/\tau^2, \kappa_2 = 0$
$\mathcal{C}_1' = 0, \mathcal{C}_2' = 0$		$m \neq 0$		$\mathcal{C}_1' = 0, \mathcal{C}_2' = 0.$

It is worth remarking that each of the above deformations introduces a new parameter κ in the initial Lie algebra and associated structures which, from the very beginning of our construction, plays the role of a *constant curvature* of some underlying symmetrical homogeneous space. In particular, in the deformation from the Galilean spacetime to the relativistic one, this process can alternatively be seen as the introduction of a *fundamental scale* c ($\kappa_2 = -1/c^2$). The sequence of these classical deformations has a natural end when the semisimple Lie algebras $\mathfrak{so}(p, q)$ are reached. In this respect, we would like to stress that the aforementioned geometrical and physical ideas behind these constructions can also be found in the framework of quantum algebras or q-deformations [31] $U_z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. These also make use of its universal enveloping algebra $U(\mathfrak{g})$ together with a deformation parameter z ($q = \mathrm{e}^z$) and the classical limit $z \to 0$ leads to $U(\mathfrak{g})$. By one hand, quantum deformations of the Poincaré algebra have been used to obtain the so-called doubly special relativities (see [32, 33] and references therein) for which the deformation parameter z is interpreted as a second *fundamental scale* related to the Planck length; thus this process is 'similar' to the introduction of c through a Galilean deformation but obviously goes beyond Lie structures. On the other hand, quantum algebras have recently been shown to provide spaces of generically *variable curvature* which is governed by z [34]. Hence when classical and quantum deformations are considered in a global picture one finds a natural 'common' deformation setting which indicates the relevance of the use of universal enveloping algebras.

Acknowledgments

References

[1] Inönü E and Wigner E P 1953 *Proc. Natl. Acad. Sci. USA* **39** 510
      Inönü E and Wigner E P 1954 *Proc. Natl. Acad. Sci. USA* **40** 119
[2] Segal I E 1951 *Duke Math. J.* **18** 221
[3] Saletan E J 1961 *J. Math. Phys.* **2** 1
[4] Moody R V and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 2227
[5] de Montigny M and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 525
[6] Weimar-Woods E 1991 *J. Math. Phys.* **32** 2028
      Weimar-Woods E 1995 *J. Math. Phys.* **36** 4519
[7] Gilmore R 1974 *Lie Groups, Lie Algebras and Some of Their Applications* (New York: Wiley)
[8] Gerstenhaber M 1964 *Ann. Math.* **79** 59
      Gerstenhaber M 1966 *Ann. Math.* **84** 1
      Gerstenhaber M 1968 *Ann. Math.* **88** 1
      Gerstenhaber M 1974 *Ann. Math.* **99** 257
[9] Rosen J and Roman P 1966 *J. Math. Phys.* **7** 2072
[10] Levy-Nahas M 1967 *J. Math. Phys.* **8** 1211
[11] Nijenhuis A and Richardson R W 1967 *J. Math. Mech.* **17** 89
[12] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1977 *Lett. Math. Phys.* **1** 521
      Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys.* **111** 61
      Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys.* **111** 111
[13] Wolf K B and Boyer C P 1974 *J. Math. Phys.* **15** 2096
[14] Herranz F J and Santander M 1999 *J. Phys. A: Math. Gen.* **32** 3743
[15] Lévy-Leblond J M 1963 *J. Math. Phys.* **4** 776
      Lévy-Leblond J M 1971 Galilei group and Galilean invariance *Group Theory and its Applications* ed E M Loebl
          (New York: Academic)
[16] Bacry H and Lévy-Leblond J M 1968 *J. Math. Phys.* **9** 1605
[17] Bacry H and Nuyts J 1986 *J. Math. Phys.* **27** 2455
[18] de Montigny M, Patera J and Tolar J 1994 *J. Math. Phys.* **35** 405
[19] Herranz F J and Santander M 1996 *J. Phys. A: Math. Gen.* **29** 6643
[20] Figueroa-O'Farrill J M 1989 *J. Math. Phys.* **30** 2735
[21] Fialowski A and de Montigny M 2005 *J. Phys. A: Math. Gen.* **38** 6335
[22] Herranz F J and Santander M 1997 *J. Phys. A: Math. Gen.* **30** 5411
[23] Jordan C 1961–1964 *Essai sur la géométrie à n dimensions, Oeuvres* (París: Gauthier–Villars)
[24] Herranz F J and Santander M 1998 *Geometría y Física* vol XXXII ed J F Cariñena, E Martínez and M F Rañada
          (Madrid: Real Academia de Ciencias) p 59 *Preprint* physics/9702030
[25] Kirillov A A 2004 *Lectures on the Orbit Method (Grad. Stud. Math.* vol 64*)* (Providence, RI: American
          Mathematical Society)
[26] Leutwyler H and Stern J 1978 *Ann. Phys., NY* **112** 392
[27] Bacry H 1967 *Leçons sur la théorie des groupes et les symétries des particules élémentaires* (Paris: Dunod)
          (English translation, London: Gordon and Breach
[28] Bacry H 1988 *Ann. Inst. Henri Poincaré* **49** 245
      Bacry H 1993 *J. Phys. A: Math. Gen.* **26** 5413
[29] Aldrovandi R, Barbosa A L, Crispino L C B and Pereida J G 1999 *Class. Quantum Grav.* **16** 495
[30] Nieto L M, Negro J and Santander M 1997 *Int. J. Mod. Phys.* A **12** 259
[31] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[32] Lukierski J and Nowicki A 2003 *Int. J. Mod. Phys.* A **18** 7
[33] Ahluwalia-Khalilova D V 2005 *Class. Quantum Grav.* **22** 1433
[34] Ballesteros A, Herranz F J and Ragnisco O 2005 *Phys. Lett.* B **610** 107
      Ballesteros A, Herranz F J and Ragnisco O 2005 *J. Phys. A: Math. Gen.* **38** 7129